Problem Set 2

Macroeconomics III

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Problem 1

Social planner's problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t (1+n)^t u(c_t)$$
$$\text{s.t.} \quad (1+n)k_{t+1} = (1-\delta)k_t + f(k_t) - c_t$$
$$k_0 > 0 \text{ given}, \quad k_{t+1} \geq 0, \quad \beta(1+n) < 1$$

Variables:

- k_t, capital per worker
- c_t, consumption per worker
- $f(k_t)$, production per worker

Parameters:

- β, discount rate
- n, population growth
- δ, capital depreciation

1. Solve the social planner's problem, find FOC and the Euler equation.
2. Characterize the steady state, draw phase diagram and characterize the dynamics.
3. Investigate dynamics in light of a shock to δ.

Problem 1a - Lagrange

Write the Lagrangian for the optimization problem, find the first order conditions that characterize optimal choices, and from these the Euler equation. Give an economic interpretation to this equation.

Step 1: Set-up the Lagrangian ▸ Two methods

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t (1+n)^t \Big[u(c_t) + \lambda_t \big((1-\delta) k_t + f(k_t) - c_t - (1+n) k_{t+1} \big) \Big]$$

Useful way to think about the Lagrangian when finding first order conditions:

$$\begin{aligned}
\mathcal{L} =& u(c_0) + \lambda_0 \big((1-\delta) k_0 + f(k_0) - c_0 - (1+n) k_1 \big) \\
& + \beta(1+n) \Big[u(c_1) + \lambda_1 \big((1-\delta) k_1 + f(k_2) - c_1 - (1+n) k_2 \big) \Big] \\
& \dots \\
& + \beta^t (1+n)^t \Big[u(c_t) + \lambda_t \big((1-\delta) k_t + f(k_t) - c_t - (1+n) k_{t+1} \big) \Big] \\
& + \beta^{t+1} (1+n)^{t+1} \Big[u(c_{t+1}) + \lambda_{t+1} \big((1-\delta) k_{t+1} + f(k_{t+1}) - c_{t+1} - (1+n) k_{t+2} \big) \Big] \\
& \dots
\end{aligned}$$

Problem 1a - FOC

FOC wrt. consumption is relatively straight forward:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \quad \implies \quad \beta^t (1+n)^t u'(c_t) = \beta^t (1+n)^t \lambda_t$$

$$u'(c_t) = \lambda_t$$

FOC wrt. capital is more tricky since it enters in two periods:

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \implies \beta^t (1+n)^t \lambda_t (1+n) = \beta^{t+1} (1+n)^{t+1} \beta \lambda_{t+1} [1 - \delta + f'(k_{t+1})]$$

$$\beta^t (1+n)^{t+1} \lambda_t = \beta^{t+1} (1+n)^{t+1} \beta \lambda_{t+1} [1 - \delta + f'(k_{t+1})]$$

$$\lambda_t = \beta \lambda_{t+1} [1 - \delta + f'(k_{t+1})]$$

Remember the Lagrangian:

$$\begin{aligned}
\mathcal{L} =& u(c_0) + \lambda_0 \big((1-\delta) k_0 + f(k_0) - c_0 - (1+n) k_1 \big) \\
& \dots \\
& + \beta^t (1+n)^t \Big[u(c_t) + \lambda_t \big((1-\delta) k_t + f(k_t) - c_t - (1+n) k_{t+1} \big) \Big] \\
& + \beta^{t+1} (1+n)^{t+1} \Big[u(c_{t+1}) + \lambda_{t+1} \big((1-\delta) k_{t+1} + f(k_{t+1}) - c_{t+1} - (1+n) k_{t+2} \big) \Big] \\
& \dots
\end{aligned}$$

Problem 1a - Euler equation

We have the FOCs, which describes the economy (wrt. c_t, k_{t+1} and λ_t):

$$u'(c_t) = \lambda_t \tag{1}$$
$$\lambda_t = \beta\lambda_{t+1}[1-\delta+f'(k_{t+1})] \tag{2}$$
$$(1+n)k_{t+1} = (1-\delta)k_t + f(k_t) - c_t \tag{3}$$

We know that the FOCs hold in every period such that: $u'(c_{t+1}) = \lambda_{t+1}$. Combine equation 1 with equation 2:

$$\begin{aligned} u'(c_t) &= \lambda_t \\ u'(c_t) &= \beta\lambda_{t+1}[1-\delta+f'(k_{t+1})] \\ u'(c_t) &= \beta u'(c_{t+1})[1-\delta+f'(k_{t+1})] \quad (4) \end{aligned}$$

Thus, we get the Euler equation:

$$\underbrace{\frac{\beta u'(c_{t+1})}{u'(c_t)}}_{MRS} = \underbrace{\frac{1}{[1-\delta+f'(k_{t+1})]}}_{\text{Relative price of consumption}} \qquad \text{(Euler equation)}$$

Marginal utility today should be equal to the future value of the discounted marginal utility of tomorrow.

Problem 1b - Steady state

Characterize the steady state, draw the dynamics of the economy in a phase diagram. Describe how the variables move, plot the saddle path. How is initial consumption determined for each possible level of k0?

We know find the conditions for which the variables are constant $c_t = c_{t+1} = c$, $k_t = k_{t+1} = k$ and $\lambda_t = \lambda_{t+1} = \lambda$. Hence, equation 4 and equation 3 become:

$$1 = \beta[(1-\delta) + f'(k)] \qquad \text{(c-locus)}$$

$$c = f(k) - (n+\delta)k \qquad \text{(k-locus)}$$

At steady state both of these equations will hold. We can then draw a curve where $c_t = c_{t+1}$ and a curve where $k_t = k_{t+1}$. Both $c_t = c_{t+1}$ and $k_t = k_{t+1}$ will be satisfied at the intersection(s).

The c-locus does not depend on c, so it will be a vertical curve at

$$f'(k^*) = \frac{1}{\beta} - (1-\delta)$$

Where k^* is the steady state level of capital. The k-locus is a concave curve that will cross the x-axis eventually.

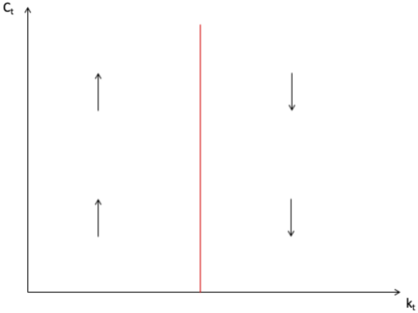
Problem 1b - C-locus

Next we want to understand what happens to c_t, when it is not on the c-locus. From equation 4, we get the following when $c_{t+1} > c_t$:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta[1 - \delta + f'(k_{t+1})] > 1$$

$$f'(k_{t+1}) > \frac{1}{\beta} - (1 - \delta) = f'(k^*) \quad \xRightarrow[\text{concave } f]{} \quad k_{t+1} < k^*$$

Hence, consumption is increasing when $k_{t+1} < k^*$ and decreasing when $k_{t+1} > k_t$. This leads us to following dynamics of c:



Low capital implies larger marginal product, which leads to higher returns to savings $\Rightarrow$

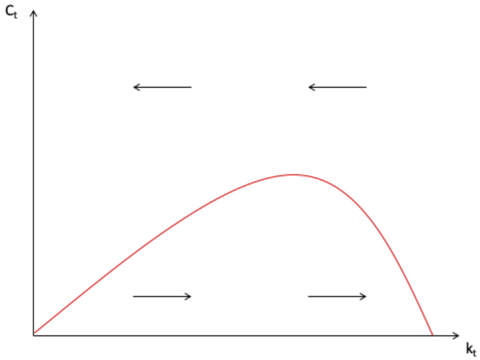
Consumption today becomes less attractive.

Problem 1b - K-locus

To investigate the dynamics of capital, we focus on the capital law of motion. Suppose $k_{t+1} > k_t$:

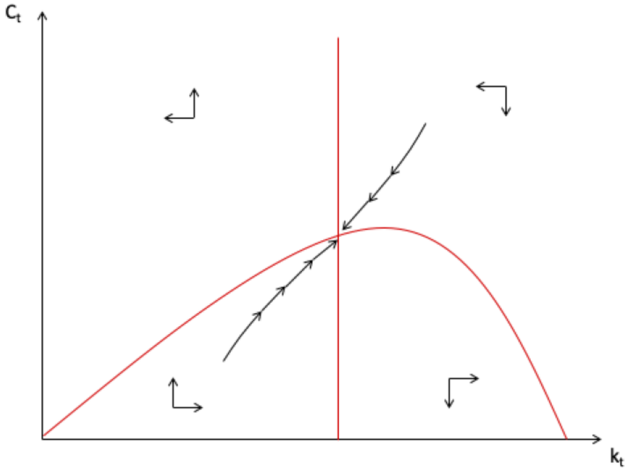
$$c_t = f(k_t) + (1-\delta)k_t - (1+n)k_{t+1} < f(k^*) - (\delta + n)k^*$$

Hence, $k_{t+1} > k_t$ when c_t is below the k-locus such that:



- Low consumption implies larger amount of investments, which leads to capital accumulation.
- Conversely, high consumption implies few investments. Capital per worker will, thus, decrease due to depreciation and population growth.

Problem 1b - Phase diagram



Note: The line with arrows is called the saddle path.

The economy will converge towards (k^*, c^*) along the saddle path. If the economy is not on the saddle path, the economy will diverge.

If the economy is not on the saddle path, one of two scenarios will occur:

- If consumption is above the saddle path, capital will decrease. This will lead to consumption growth as interest rates increase, which will further decrease capital accumulation. Ultimately, agents will consume everything.
- If consumption is below the saddle path, capital will increase. This will lower interest rates, which further lowers consumption (see equation 4) and increases capital accumulation. Ultimately, agents will invest everything.

The intuition is hardly intuitive but it might help to look at the Euler equation as it is largely based on that.

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + f'(k_{t+1})]$$

Problem 1c

The economy is in a steady state and there is an unexpected permanent increase in the rate of depreciation δ. What is the best response to this change? Does consumption initially increases or decreases?

We find the effect on each loci by differentiating wrt. δ. Differentiating the c-locus, yields:

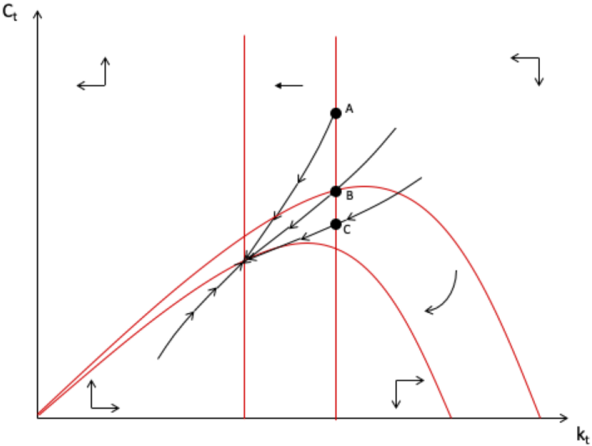
$$0 = \beta\left[(0-1) + \frac{\partial f'(k)}{\partial \delta}\right]$$
$$1 = \frac{\partial f'(k)}{\partial \delta} = f''(k)\frac{dk}{d\delta} \implies \frac{dk}{d\delta} = \frac{1}{f''(k)} < 0$$

Since f is concave, $f'(k*)$ is decreasing in k^*, which implies that the c-locus moves to the left. Differentiating the k-locus yields:

$$\frac{\partial c}{\partial \delta} = -k < 0$$

Hence, the k-locus decreases.

Problem 1c - Illustration



Substitution effect (A): When δ increases, gross interests decrease and savings become less attractive $\implies$ Consumption should increase on impact.

Income effect (C): When gross interests decrease, the value of savings decrease, so households need to save more $\implies$ Consumption should decrease on impact.

Problem 1c - Effect on impact

Suppose the agents have CRRA utility with risk aversion parameter θ:

- If $\theta < 1$: The substitution effect will dominate so household will increase consumption on impact as a response to the higher depreciation rate of capital. Hence, consumption will jump to point A.
- If $\theta = 1$: The substitution and income effect will cancel each other, which implies that the economy will jump to point B.
- If $\theta > 1$: The income effect will dominate, since households will dislike fluctuations in the consumption level. The economy will jump to point C.

The short-run effect depends on the functional form of the utility function of the household.

Problem 2

Firms face the following static problem:

$$\max_{L_t, K_t} \pi(K_t, L_t) = AK_t^\alpha L_t^{1-\alpha} - w_t L_t - r_t K_t$$

Households face the following problem:

$$\max_{c_t, a_{t+1}} U = \sum_{t=0}^{\infty} \beta^t (1+n)^t \frac{c_t^{1-\sigma}}{1-\sigma}, \quad \sigma > 0, \quad \sigma \neq 1$$
$$\text{s.t. } (1+n)a_{t+1} = a_t R_t + w_t - c_t$$

Further, we know that:

$$a_t = k_t + b_{pt}$$
$$R_t = 1 + r_t - \delta$$

Problem 2a - Profit maximization and NPGC

Firms problem: The problem of the firm is static, which simplifies the derivation. Finding first order conditions of the profit function wrt. L_t and K_t yields:

$$\text{FOC wrt. } L_t: \quad w_t = A(1-\alpha)K^{\alpha}L^{-\alpha} = A(1-\alpha)k_t^{\alpha} \tag{5}$$

$$\text{FOC wrt. } K_t: \quad r_t = A\alpha K^{\alpha-1}L^{1-\alpha} = A\alpha k_t^{\alpha-1} \tag{6}$$

The wage rate must be equal to the marginal product of labour and the interest rate must be equal to the marginal product of capital.

NPGC: The No Ponzi-Game Condition eliminates the households' ability to roll over debt indefinitely. The NPGC is given by:

$$\lim_{T\to\infty} q_T a_{T+1} \geq 0 \tag{NPGC}$$

Where $q_t = \prod_{i=1}^{t} \frac{(1+n)}{R_i}$.

Problem 2a - Langrangian

The (current-value) Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t (1+n)^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \lambda_t \Big(a_t(1+r_t-\delta) + w_t - c_t - (1+n)a_{t+1} \Big) \right]$$

The first order conditions wrt c_t, a_{t+1} and λ_t respectively are:

$$c_t^{-\sigma} = \lambda_t \tag{7}$$

$$\lambda_t = \beta\lambda_{t+1}\big[1 + r_{t+1} - \delta\big] \tag{8}$$

$$(1+n)a_{t+1} = a_t(1+r_t-\delta) + w_t - c_t \tag{9}$$

Combining equation 6, 7 and 8 yields the Euler equation:

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}\big[1 + r_{t+1} - \delta\big] = \beta c_{t+1}^{-\sigma}\big[1 + A\alpha k_{t+1}^{\alpha-1} - \delta\big] \tag{10}$$

$$\frac{c_{t+1}}{c_t} = \big[\beta(1 + A\alpha k_{t+1}^{\alpha-1} - \delta)\big]^{\frac{1}{\sigma}} \tag{11}$$

Interpretation is equivalent to problem 1. The higher the amount of risk-aversion, σ, the smaller change in relative consumption from a change in the discount factors.

Problem 2b - Intertemporal budget constraint (1/2)

Show that in an optimum the no Ponzi game condition must hold with equality (this is the transversality condition for the planner's problem).

Find the intertemporal budget constraint by using the budget constraint $c_t + (1+n)a_{t+1} = a_t R_t + w_t$ recursively:

$$c_0 + (1+n)a_1 = a_0 R_0 + w_0$$
$$c_1 + (1+n)a_2 = a_1 R_1 + w_1 \implies a_1 = \frac{1}{R_1}(c_1 + (1+n)a_2 - w_1)$$

Combining the two yields:

$$c_0 + \frac{1+n}{R_1}c_1 + \frac{1+n}{R_1}(1+n)a_2 = a_0 R_0 + w_0 + \frac{1+n}{R_1}w_1 \tag{12}$$

Doing it for one more period yields:

$$c_0 + \frac{1+n}{R_1}c_1 + \frac{(1+n)^2}{R_1 R_2}c_2 + \frac{(1+n)^3}{R_1 R_2}a_3 = a_0 R_0 + w_0 + \frac{1+n}{R_1}w_1 + \frac{(1+n)^2}{R_1 R_2}w_2$$

Problem 2b - Intertemporal budget constraint (2/2)

Recursively substituting leads to:

$$c_0 + \lim_{T\to\infty} \sum_{t=1}^{T} \frac{(1+n)^t c_t}{\prod_{i=1}^{t} R_i} + (1+n) \lim_{T\to\infty} \frac{(1+n)^T a_{T+1}}{\prod_{i=1}^{T} R_i} = a_0 R_0 + w_0 + \lim_{T\to\infty} \sum_{t=1}^{T} \frac{(1+n)^t w_t}{\prod_{i=1}^{t} R_i}$$

We apply the expression for prices, $q_t = \prod_{i=1}^{t} \frac{1+n}{R_i}$ and $q_0 = 1$:

$$\lim_{T\to\infty} \sum_{t=0}^{T} q_t c_t + (1+n) \lim_{T\to\infty} q_T a_{T+1} = a_0 R_0 + \lim_{T\to\infty} \sum_{t=0}^{T} q_t w_t$$

Which can be rewritten as:

$$\lim_{T\to\infty} \sum_{t=0}^{T} q_t c_t = a_0 R_0 + \lim_{T\to\infty} \sum_{t=0}^{T} q_t w_t - (1+n) \lim_{T\to\infty} q_T a_{T+1}$$

Problem 2b - NPGC and TVC

Ultimately, the intertemporal budget constraint becomes:

$$\lim_{T\to\infty}\sum_{t=0}^{T} q_t c_t = a_0 R_0 + \lim_{T\to\infty}\sum_{t=0}^{T} q_t w_t - \underbrace{(1+n)\lim_{T\to\infty} q_T a_{T+1}}_{\geq 0 \text{ By the NPCG}}$$

Note: The left hand side is the net present value of consumption. In order to maximize the left hand side, the term $\lim_{T\to\infty} q_T a_{T+1}$ should be as small as possible.

The No Ponzi Game Condition states:

$$\lim_{T\to\infty} q_T a_{T+1} \geq 0$$

Hence, the present value of consumption is maximized when the term is zero and the NPGC holds with equality such that

$$\lim_{T\to\infty} q_T a_{T+1} = 0$$

Therefore, the NPGC will hold with equality.

Problem 2c - Initial consumption (1/3)

Using the intertemporal budget constraint find initial consumption for the special case of logarithmic preferences ($\sigma = 1$). Interpret.

To find the initial consumption, we iterate the Euler equation:

$$c_1 = c_0[\beta R_1]^{\frac{1}{\sigma}} \underbrace{=}_{\sigma=1} c_0 \beta R_1 \tag{13}$$

We then rewrite the budget constraint from equation 12:

$$c_0 + \frac{1+n}{R_1} c_1 = a_0 R_0 + w_0 + \frac{1+n}{R_1} w_1 - \frac{(1+n)^2}{R_1} a_2$$

Inserting equation 13 into the left hand side implies:

$$c_0 + \frac{1+n}{R_1} c_1 = c_0 + (1+n)\beta c_0 = c_0(1 + (1+n)\beta)$$

Iterating forward and remembering that the TVC holds leads to:

$$c_0 \sum_{t=0}^{\infty} [\beta(1+n)]^t = a_0 R_0 + \sum_{t=0}^{\infty} q_t w_t \tag{14}$$

Problem 2c - Initial consumption (2/3)

We rewrite the left hand side of the equation since it is a geometric series:

$$c_0 \sum_{t=0}^{\infty} [\beta(1+n)]^t = \frac{c_0}{1-\beta(1+n)}$$

Isolating c_0 in equation 14 is then straightforward:

$$c_0 = (1-\beta(1+n)) \left[a_0 R_0 + \sum_{t=0}^{\infty} q_t w_t \right] \tag{15}$$

We can define, the net present value of labour income as

$$W_0 = \sum_{t=0}^{\infty} q_t w_t$$

The initial consumption is then:

$$c_0 = (1-\beta(1+n)) \left[a_0 R_0 + W_0 \right] \tag{16}$$

Problem 2c - Initial consumption (3/3)

$$c_0 = (1 - \beta(1+n)) \underbrace{\left[a_0 R_0 + W_0\right]}_{\text{Total wealth}} \tag{17}$$

Increase in β or n leads to a decrease in c_0. *What is the intuition?*

- If the discount factor, β, increases, agents become more patient. Hence, they will lower their initial consumption as they will value consumption in future periods more than before the increase in β.
- Equivalently, if the household/population is growing faster (increase in n), then the initial consumption will decrease as well.
- Assuming that W_0 is fixed, the initial consumption does not depend on future interest rates since we have assumed log-preferences such that income and substitution effects offset each other.
- Interest rates can, however, affect W_0 in which case it is a wealth effect - this is not present if we assume fixed W_0.

Problem 2d - c-locus and k-locus

Draw the phase diagram and find the equations that characterize steady state. What is the effect of a permanent increase in n? Given this, discuss the model's implications for capital accumulation in countries with high or low fertility.

We note that in equilibrium there will be zero net borrowing $\implies b_{pt} = 0$ and thus $a_t = k_t$.

This implies that the equations governing the dynamics of the model are:

$$u'(c_t) = \beta u'(c_{t+1})\left[1 - \delta + A\alpha k_{t+1}^{\alpha-1}\right]$$
$$(1+n)k_{t+1} = (1-\delta)k_t + Ak^{\alpha} - c_t$$

These resemble those of problem 1, why the loci are almost identical:

$$c = Ak^{\alpha} - (n+\delta)k \qquad \text{(k-locus)}$$
$$1 = \beta\left[1 - \delta + A\alpha k_t^{\alpha-1}\right] \qquad \text{(c-locus)}$$

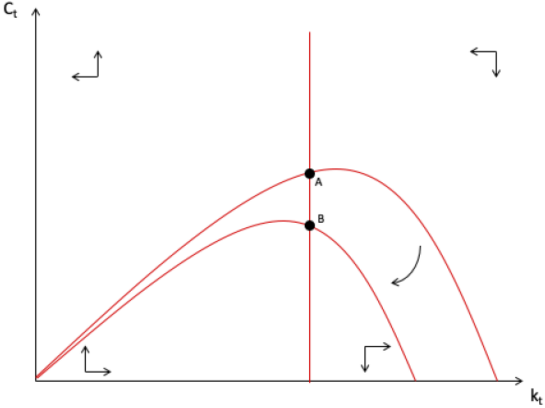
The form of the phase diagram is identical to the previous one.

Problem 2d - increase in n

By differentiating wrt. n, it becomes evident that n merely affects c:

$$\frac{\partial c}{\partial n} = -k$$

k is determined from the c-locus, why n does not affect k. An increase in n moves the k-locus so the economy moves from point A to B:



- Consumption decreases on impact. c_t captures the full adjustment such that capital accumulation is unchanged.
- This result implies that fertility rate does not affect capital per capita but consumption per capita is decreasing in the fertility rate.

Lagrangian

Two types of Lagrangian - Present-value (PV) and current-value (CV):

$$\mathcal{L} = \sum_{t=0}^{\infty} \left[\beta^t (1+n)^t u(c_t) + \phi_t \big((1-\delta) k_t + f(k_t) - c_t - (1+n) k_{t+1} \big) \right] \text{ (PV)}$$

ϕ_t is the value of a relaxation of the constraint at t time but in units of time zero utility. The utility increase at time t is discounted such that it is the present value of such a relaxation of the constraint.

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t (1+n)^t \left[u(c_t) + \lambda_t \big((1-\delta) k_t + f(k_t) - c_t - (1+n) k_{t+1} \big) \right] \text{ (CV)}$$

λ_t represents the value in utility at time t following a relaxation of the constraint at time t. Hence, it is the current value (value at time t) of a relaxation of the constraint.

$\longrightarrow$ The solutions are identical but different interpretations of ϕ_t and λ_t.

Current-value Lagrangians are often simpler and easier to interpret. Online resources often focus on the PV and CV Hamiltonian. The Hamiltonian is the continuous time version of the Lagrangian. Back